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A HIGH ORDER GENERALIZED METHOD OF AVERAGING*

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Abstract. We develop a high order generalized perturbation technique that extends the Krylov–Bogoliubov–Mitropolsky method of averaging to vector systems written in normal form with multiple angular components. An algorithm is presented that iteratively gives the terms in the asymptotic approximation. A nonresonance condition is assumed that guarantees the smoothness of the terms. The main result establishes that the absolute error between the unaveraged normal system and its Nth order approximation is of the order of the Nth power of the perturbation parameter for a time interval of length the order of the reciprocal of the perturbation parameter. The high order algorithm is applied to a coupled van der Pol oscillator system. Some numerical results are given to show that the main result reflects actual computational experience.

1. Introduction. One of the techniques used to study periodic phenomena associated with nonlinear mechanical systems is to reduce the study of the describing differential equations to a standard form

$$\dot{x} = \varepsilon X(t, x), \qquad x(0, \varepsilon) = x_0, \tag{1.1}$$

where $x \in R^m$. The method of integral averaging of Bogoliubov and Mitropolsky [7] can then be applied to put (1.1) into a form

$$\dot{x} = \varepsilon X_0(x), \tag{1.2}$$

where

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t, x)\, dt. \tag{1.3}$$

The qualitative behavior of the solutions of (1.1) can be studied through (1.2).

Higher order methods of averaging have been studied by several authors. Perko [30] extended the averaging procedure for (1.1) to an Nth order result. In particular, he showed that there exists a transformation of the form

$$x = y + \sum_{j=1}^{N} \varepsilon^j v_j(t, y) \tag{1.4}$$

that transforms (1.1) into an equivalent form

$$\dot{y} = \sum_{j=1}^{N} \varepsilon^j f_j(y) + \varepsilon^{N+1} f_{N+1}(t, y, \varepsilon). \tag{1.5}$$

He develops a specific term by term algorithm to generate v_j, f_j, and establishes an approximation result that shows that the solution of (1.1) and a certain transformed solution of

$$\dot{y} = \sum_{j=1}^{N} \varepsilon^j f_j(y), \tag{1.6}$$

with the appropriate initial condition, compare to within a power N of the small parameter ε. Volosov [35] considered the same question, but after giving a procedure for the first two terms did not give a full term by term algorithm to general order.

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Zabreiko and Ledovskaja [36] do give a term by term algorithm, although in a somewhat different form from Perko. They also state but do not prove an Nth order approximation theorem. Morrison [22] developed an averaging algorithm to the second order for (1.1).

When one is dealing with weakly perturbed oscillators of the form

$$\ddot{z}_j + \omega_j^2 z_j = \varepsilon Z_j(z, \dot{z}), \tag{1.7}$$

$j = 1, \cdots, m$, $\omega_j > 0$, where $z = (z_1, \cdots, z_m)$, $z = (\dot{z}_1, \cdots, \dot{z}_m)$, it is sometimes more appropriate to introduce polar type coordinates of the form, β real,

$$z_j = x_j^\beta \sin \omega_j\theta_j, \qquad \dot{z}_j = x_j^\beta \omega_j \cos \omega_j\theta_j, \tag{1.8}$$

$j = 1, 2, \cdots, m$. Then (1.7) reduces to a system of the form

$$\dot{\theta} = d + \varepsilon\Theta(\theta, x), \qquad \dot{x} = \varepsilon X(\theta, x), \tag{1.9}$$

where $d = \mathrm{col}(1, 1, \cdots, 1)$, $x = \mathrm{col}(x_1, \cdots, x_m)$, $\theta = \mathrm{col}(\theta_1, \cdots, \theta_m)$. (1.9) is said to be in normal form.

In [14] the author extended the Bogoliubov and Mitropolsky [7] averaging result to (1.9), and developed a comparison theorem between (1.9) and

$$\dot{\theta} = d + \varepsilon\Theta_0(x), \qquad \dot{x} = \varepsilon X_0(x), \tag{1.10}$$

where Θ_0, X_0 are averages of the form

$$f_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\theta + s, x)\, ds, \tag{1.11}$$

$\theta + s = (\theta_1 + s, \cdots, \theta_m + s)$. This form of averaging was previously used by Diliberto [12]. Formal properties of this method of averaging have been studied by Kirchgraber [17]. These generalized averaging procedures have been used in orbital calculations by Velez and Fuchs [34] and in nearly Hamiltonian systems of two degrees of freedom by Murdock [25]. Sethna and Schapiro [31] have applied the results in [14] to flutter unstable dynamical systems. Another first order generalized averaging theorem has also been stated in Arnold and Avez [3]. Giacaglia [13] and Hale [15] relate this averaging principle to the study of stability properties of dynamical systems near invariant manifolds.

The essence of the method of averaging rests upon introducing a near-identity transformation into (1.9) that reduces it to a system that is a high order perturbation of (1.10). For an overview of the use of near-identity transformations see Neu [29]. Near-identity transformations represent a method of introducing local coordinates around periodic solutions. This idea of using local coordinates to decompose systems has been extended to functional differential equations by Stokes [33].

In the present paper the author extends the result of Perko [30] to systems of the form (1.9). In particular, (1.9) is transformed by

$$\theta = \phi + \sum_{j=1}^{N} \varepsilon^j u_j(\phi, r), \qquad x = r + \sum_{j=1}^{N} \varepsilon^j w_j(\phi, r), \tag{1.12}$$

to a system that is a high order perturbation of

$$\dot{\phi} = d + \sum_{j=1}^{N} \varepsilon^j \Phi_j(r), \qquad \dot{r} = \sum_{j=1}^{N} \varepsilon^j R_j(r). \tag{1.13}$$

Explicit expressions for computing (1.13) up to second order terms have been given by several authors. See, e.g., Bajaj, Sethna and Lundgren [4], Morrison [23] and Nayfeh [27]. Musen [26] has developed a formal procedure that is analogous to the methods of this paper. His approach is by way of certain formal operators. No proofs are given, though, on the nature of the approximation obtained. A somewhat different approach to higher order perturbations was taken by Agrawal and Evan-Iwanowski [1]. They approached system (1.7) and applied the Bogoliubov and Mitropolsky [7] perturbation technique directly to obtain higher order approximations. They derive the subsidiary equations that must be solved to obtain the perturbation terms but do not explicitly solve them. Finally, Chow and Mallet-Paret [9], [10] develop a different high order scheme but do not treat the problem of multiple angles as developed in this paper.

In § 3 the algorithm is given to compute Φ_i, R_i, u_i, w_i. Two lemmas are also proven that establish the differentiability properties of these functions. The main result of this paper is Theorem 4.1, in § 4. The proof follows exactly the lines of the proof in Perko [30, Thm. 1]. Perko's argument, however, has been modified by the author to take into account the appearance of multiple angles in (1.9). All of these results depend on the generalized notion of derivatives of vector valued functions of several vector variables. This is discussed in § 2, and the necessary algebraic relations are given. These derivatives are nothing more than the Fréchet derivatives specialized to finite dimensional space. Finally, in § 5 the results are applied to a weakly coupled system of van der Pol oscillators.

2. Notation. Let R^m be an m-dimensional real Euclidean space, $G^m \subset R^m$ compact and convex and $\Sigma = R^m \times G^m$. Let $f(\theta, x) \in P^\alpha_\omega(\Sigma)$, for $(\theta, x) \in \Sigma$, if f is continuously differentiable in θ, x, up to order α and periodic in 0 with vector period $2\pi/\omega = (2\pi/\omega_1, \cdots, 2\pi/\omega_m)$.

For (θ, x), $(h, k) \in \Sigma$, $f \in P^\alpha_\omega(\Sigma)$ define (see Bartle [5]) the Fréchet derivative

$$D_1 f(\theta, x)\cdot h = \left(\frac{\partial f_i}{\partial \theta_j}\right)_{i,j=1,m} (h_j)_{j=1,m}, \tag{2.1}$$

and similarly for $D_2 f(\theta, x)\cdot k$. Then, following Dieudonné [11], define

$$Df(\theta, x)\cdot(h, k) = D_1 f(\theta, x)\cdot h + D_2 f(\theta, x)\cdot k, \tag{2.2}$$

and, for $n > 0$,

$$\begin{aligned} &D^n f(\theta, x)\cdot(h_1, k_1)\cdots(h_n, k_n) \\ &\quad = D(D^{n-1} f(\theta, x)\cdot(h_1, k_1)\cdots(h_{n-1}, k_{n-1}))\cdot(h_n, k_n). \end{aligned} \tag{2.3}$$

If $h_1 = \cdots = h_n = h$, $k_1 = \cdots = k_n = k$, set

$$(h, k)^n = (h, k)\cdots(h, k) \qquad (n \text{ times}), \tag{2.4}$$

and

$$D^n f(\theta, x)\cdot(h, k)^n = D(D^{n-1} f(\theta, x)\cdot(h, k)^{n-1})(h, k). \tag{2.5}$$

If $\alpha \geqq N+1$, $(\theta, x) \in \Sigma$, the Taylor series becomes

$$\begin{aligned} f(\theta + h, x + k) &= f(\theta, x) + Df(\theta, x)\cdot(h, k) + \cdots \\ &\quad + \frac{1}{N!} D^N f(\theta, x)\cdot(h, k)^N + R_N(\theta, x), \end{aligned} \tag{2.6}$$

where

$$R_N(\theta, x) = \left(\frac{1}{N!}\int_0^1 (1-\gamma)^N D^{N+1} f(\theta+\gamma h, x+\gamma k)\, d\gamma\right) \cdot (h, k)^{N+1}. \tag{2.7}$$

For the derivation of the Taylor series above see either Dieudonné [11] or Liusternik and Sobolev [20].

The definition has some algebraic consequences that will be needed in later sections. Let $(\theta, x) \in \Sigma$, $f \in P^{\alpha}_{\omega}(\Sigma)$;

$$D\left(\sum_{i=1}^{M} a_i f_i(\theta, x)\right) \cdot \left(\sum_{j=1}^{N} b_j h_j, \sum_{j=1}^{N} b_j k_j\right) = \sum_{i=1}^{M}\sum_{j=1}^{N} a_i b_j D f_i(\theta, x) \cdot (h_j, k_j), \tag{2.8}$$

where $M, N > 0$, and a_i, b_j are constants. If $k \geqq 1$ then, by induction on k and rules of powers of polynomial forms,

$$\begin{aligned} &D^k f(\theta, x) \cdot \left(\sum_{j=1}^{N} \varepsilon^j u_j, \sum_{j=1}^{N} \varepsilon^j w_j\right)^k \\ &\qquad = D^k f(\theta, x) \cdot \sum_{i=1}^{kN} \varepsilon^i \sum_{j_1+\cdots+j_k=i} (u_{j_1}, w_{j_1}) \cdots (u_{j_k}, w_{j_k}), \end{aligned} \tag{2.9}$$

where $1 \leqq j_1, \cdots, j_k \leqq N$. (See, e.g., Liusternik and Sobolev [20]). Also, by induction on k,

$$\begin{aligned} &D^k f(\theta, x) \cdot \left(\sum_{i=1}^{N} \varepsilon^i u_i + \varepsilon^{N+1} U_{N+1}, \sum_{i=1}^{N} \varepsilon^i w_i + \varepsilon^{N+1} W_{N+1}\right)^k \\ &\qquad = D^k f(\theta, x) \cdot \left(\sum_{i=1}^{N} \varepsilon^i u_i, \sum_{i=1}^{N} \varepsilon^i w_i\right)^k + \varepsilon^{N+1} R_{N+1}, \end{aligned} \tag{2.10}$$

where U_{N+1}, W_{N+1}, R_{N+1} are error terms.

Let $f \in P^{\alpha}_{\omega}(\Sigma)$, for some $\alpha > 0$. Define the *mean value* of f by the relation

$$M_\theta f = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\theta+s, x)\, ds, \tag{2.11}$$

where $\theta + s = (\theta_1 + s, \cdots, \theta_m + s)$. A sufficient condition for the limit in (2.11) to exist independent of θ is that the frequencies $\omega_1, \cdots, \omega_m$ of $f(\theta, x)$ with respect to θ be linearly independent over the integers. In this case (2.11) can be replaced by

$$M_\theta f = \frac{\omega_1 \cdots \omega_m}{(2\pi)^m}\int_0^{2\pi/\omega_1} \cdots \int_0^{2\pi/\omega_m} f(\theta, x)\, d\theta. \tag{2.12}$$

For a proof of this result see Arnold [2]. From a physical point of view, imposing this independence requirement amounts to restricting oneself to nonresonant mechanical oscillations. In fact the methods developed in this paper are applicable to systems generating self-sustaining periodic oscillations.

3. The averaging algorithm. Let Θ, $X \in P^{\alpha}_{\omega}(\Sigma)$, and

$$\begin{aligned} \dot\theta &= d + \varepsilon\Theta(\theta, x), \qquad \theta(0) = \theta_0, \\ \dot x &= \varepsilon X(\theta, x), \qquad x(0) = x_0, \end{aligned} \tag{3.1}$$

where $d=(1,\cdots,1)$. To average (3.1), a near-identity transformation of the form

$$\theta=\phi+\sum_{i=1}^{N}\varepsilon^i u_i(\phi,r),\qquad x=r+\sum_{i=1}^{N}\varepsilon^i w_i(\phi,r) \tag{3.2}$$

is sought that reduces (3.1) to

$$\begin{aligned}\dot\phi&=d+\sum_{i=1}^{N}\varepsilon^i\Phi_i(r)+\varepsilon^{N+1}\Phi_{N+1}(\phi,r,\varepsilon),\\ \dot r&=\sum_{i=1}^{N}\varepsilon^i R_i(r)+\varepsilon^{N+1}R_{N+1}(\phi,r,\varepsilon),\end{aligned} \tag{3.3}$$

where u_i, w_i, Φ_i, R_i are to be determined. The initial conditions for (3.3) are implicitly determined by

$$\begin{aligned}\phi(0,\varepsilon)&=\theta_0-\sum_{i=1}^{N}\varepsilon^i u_i(\phi(0,\varepsilon),r(0,\varepsilon))+O(\varepsilon^{N+1}),\\ r(0,\varepsilon)&=x_0-\sum_{i=1}^{N}\varepsilon^i w_i(\phi(0,\varepsilon),r(0,\varepsilon))+O(\varepsilon^{N+1}).\end{aligned} \tag{3.4}$$

The computations in this section are generalizations of those given in Perko [30].

Insert (3.2) into the left-hand side of (3.1) and get

$$\begin{aligned}\dot\theta&=\dot\phi+\sum_{i=1}^{N}\varepsilon^i Du_i(\phi,r)\cdot(\dot\phi,\dot r),\\ \dot x&=\dot r+\sum_{i=1}^{N}\varepsilon^i Dw_i(\phi,r)\cdot(\dot\phi,\dot r).\end{aligned} \tag{3.5}$$

From (3.3), (2.8) and (2.10), (3.5) becomes

$$\begin{aligned}\dot\theta&=\dot\phi+\sum_{i=1}^{N}\varepsilon^i D_1u_i(\phi,r)\cdot d+\sum_{i=2}^{N}\varepsilon^i\left(\sum_{j=1}^{i-1}Du_j(\phi,r)\cdot(\Phi_{i-j},R_{i-j})\right)+\varepsilon^{N+1}E_{11}(\phi,r,\varepsilon),\\ \dot x&=\dot r+\sum_{i=1}^{N}\varepsilon^i D_1W_i(\phi,r)\cdot d+\sum_{i=2}^{N}\varepsilon^i\left(\sum_{j=1}^{i-1}Dw_j(\phi,r)\cdot(\Phi_{i-j},R_{i-j})\right)+\varepsilon^{N+1}E_{12}(\phi,r,\varepsilon).\end{aligned} \tag{3.6}$$

Now expand the right-hand side of (3.1) by the Taylor series (2.6) and use (3.2), (2.9) and (2.10) to get

$$\begin{aligned}d+\varepsilon\Theta(\theta,x)&=d+\varepsilon\Theta(\phi,r)\\ &\quad+\sum_{i=2}^{N}\varepsilon^i\left(\sum_{k=1}^{i-1}\left(\frac{1}{k!}\right)D^k\Theta(\phi,r)!\cdot\sum_{j_1+\cdots+j_k=i-1}(u_{j_1},w_{j_1})\cdots(u_{j_k},w_{j_k})\right)\\ &\quad+\varepsilon^{N+1}E_{21}(\phi,r,\varepsilon),\\ \varepsilon X(\theta,x)&=\varepsilon X(\phi,r)\\ &\quad+\sum_{i=2}^{N}\varepsilon^i\left(\sum_{k=1}^{i-1}\left(\frac{1}{k!}\right)D^kX(\phi,r)\cdot\sum_{j_1+\cdots+j_k=i-1}(u_{j_1},w_{j_1})\cdots(u_{j_k},w_{j_k})\right)\\ &\quad+\varepsilon^{N+1}E_{22}(\phi,r,\varepsilon).\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) gives

$$\dot\phi = d + \varepsilon\{\Theta(\phi, r) - D_1 u_1(\phi, r)\cdot d\}$$
$$+\sum_{i=2}^{N} \varepsilon^i\Big\{\sum_{k=1}^{i-1}\Big[\Big(\frac{1}{k!}\Big)D^k\Theta(\phi, r)\cdot \sum_{j_1+\cdots+j_k=i-1}(u_{j_1}, w_{j_1})\cdots(u_{j_k}, w_{j_k})$$
$$-Du_k(\phi, r)\cdot(\Phi_{i-k}, R_{i-k})\Big] - D_1 u_i(\phi, r)\cdot d\Big\}$$
$$+\varepsilon^{N+1}E_{31}(\phi, r, \varepsilon),$$

(3.8)

$$\dot r = \varepsilon\{X(\phi, r) - D_1 W_1(\phi, r)\cdot d\}$$
$$+\sum_{i=2}^{N} \varepsilon^i\Big\{\sum_{k=1}^{i-1}\Big[\Big(\frac{1}{k!}\Big)D^k X(\phi, r)\cdot \sum_{j_1+\cdots+j_k=i-1}(u_{j_1}, w_{j_1})\cdots(u_{j_k}, w_{j_k})$$
$$-Dw_k(\phi, r)\cdot(\Phi_{i-k}, R_{i-k})\Big] - D_1 w_i(\phi, r)\cdot d\Big\}$$
$$+\varepsilon^{N+1}E_{32}(\phi, r, \varepsilon).$$

To simplify the notation define the new functions

$$F_1(\phi, r) = \Theta(\phi, r),$$
$$G_1(\phi, r) = X(\phi, r),$$

(3.9)

$$F_i(\phi, r) = \sum_{k=1}^{i-1}\Big[\Big(\frac{1}{k!}\Big)D^k\Theta(\phi, r)\cdot \sum_{j_1+\cdots+j_k=i-1}(u_{j_1}, w_{j_1})\cdots(u_{j_k}, w_{j_k}) - Du_k(\phi, r)\cdot(\Phi_{i-k}, R_{i-k})\Big],$$
$$G_i(\phi, r) = \sum_{k=1}^{i-1}\Big[\Big(\frac{1}{k!}\Big)D^k X(\phi, r)\cdot \sum_{j_1+\cdots+j_k-1}(u_{j_i}, w_{j_1})\cdots(u_{j_k}, w_{j_k}) - Dw_k(\phi, r)\cdot(\Phi_{i-k}, R_{i-k})\Big]$$

for $i = 2, \cdots, N$. If (3.3) and (3.9) are compared, the following differential equations

$$D_1 u_i(\phi, r)\cdot d = F_i(\phi, r) - \Phi_i(r), \tag{3.10a}$$
$$D_1 w_i(\phi, r)\cdot d = G_i(\phi, r) - R_i(r), \tag{3.10b}$$

$i = 1, \cdots, N$ must be solved. In order to solve (3.10) for u_i and w_i, one must make appropriate choices of the functions Φ_i and R_i for $i = 1, \cdots, N$.

Since (3.10a) and (3.10b) are formally the same, one need only solve, for example,

$$D_1 u(\phi, r)\cdot d = F(\phi, r) - \Phi(r), \tag{3.11}$$

where $F \in P^\alpha_\omega(\Sigma)$, for α sufficiently large. Subscripts have been dropped in (3.11) in order to simplify the notation somewhat.

F can be expanded as a Fourier series

$$F(\phi, r) = \sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_m=-\infty}^{\infty} F_{n_1\cdots n_m}(r)\, e^{i(n_1\omega_1\phi_1+\cdots+n_m\omega_m\phi_m)}, \tag{3.12}$$

where $i = \sqrt{-1}$ and

$$F_{n_1\cdots n_m}(r) = \Big(\frac{\omega_1\cdots\omega_m}{(2\pi)^m}\Big)\int_0^{2\pi/\omega_1}\cdots\int_0^{2\pi/\omega_m} F(\phi, r)\, e^{-i(n_1\omega_1\phi_1+\cdots+n_m\omega_m\phi_m)}\, d\phi. \tag{3.13}$$

The mean value of $F(\phi, r)$ is given by

$$M_\phi F = \left(\frac{\omega_1 \cdots \omega_m}{(2\pi)^m}\right) \int_0^{2\pi/\omega_1} \cdots \int_0^{2\pi/\omega_m} F(\phi, r)\, d\phi. \tag{3.14}$$

Define a new series

$$u(\phi, r) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_m=-\infty}^{\infty} \left(\frac{F_{n_1 \cdots n_m}(r)}{i(n, \omega)}\right) e^{i(n_1\omega_1\phi_1 + \cdots + n_m\omega_m\phi_m)}, \tag{3.15}$$

$|n| \neq 0$, where $|n| = |n_1| + \cdots + |n_m|$ and $(n, \omega) = n_1\omega_1 + \cdots + n_m\omega_m$. Suppose, furthermore, that the vector frequency $\omega = (\omega_1, \cdots, \omega_m)$ satisfies

$$|(n, \omega)| \geqq A|n|^{-(m+1)} \tag{3.16}$$

for all $n = (n_1, \cdots, n_m)$, n_i integer, $n \neq 0$, $A > 0$. (3.16) holds for almost every ω and all $n \neq 0$ (Koksma [18]).

If (3.15) and its derivative converge uniformly then a direct computation shows that u satisfies

$$D_1 u(\phi, r) \cdot d = F(\phi, r) - (M_\phi F)(r). \tag{3.17}$$

Comparing (3.17) with (3.11) indicates that one must choose

$$\Phi(r) = (M_\phi F)(r). \tag{3.18}$$

Therefore, in (3.10) one must choose

$$\Phi_i(r) = (M_\phi F_i)(r), \qquad R_i(r) = (M_\phi G_i)(r) \tag{3.19}$$

for $i = 1, 2, \cdots, N$.

LEMMA 3.1. *Let N, $k > 0$ be given, $0 \leqq n \leqq N+1$, α an even integer, $\alpha \geqq N + 2m + 3$. If Φ is defined by* (3.18), *$F \in P_\omega^\alpha(\Sigma)$, ω satisfying* (3.16), *then* (3.15) *solves* (3.17), *is uniformly convergent and differentiable to order $N+1$ and the series for $D^n u(\phi, r)$ is also uniformly convergent.*

Proof. Define the operator $\Delta_\alpha = \sum_{k=1}^m (\partial^\alpha/\partial\phi_k^\alpha)$. $F \in P_\omega^\alpha(\Sigma)$ implies there exists a $B > 0$ such that $|\Delta_\alpha F(\phi, r)| \leqq B$. By integration

$$\begin{aligned} &\left(\frac{\omega_1 \cdots \omega_m}{(2\pi)^m}\right) \int_0^{2\pi/\omega_1} \cdots \int_0^{2\pi/\omega_m} \Delta_\alpha F(\phi, r)\, e^{-i(n_1\omega_1\phi_1 + \cdots + n_m\omega_m\phi_m)}\, d\phi \\ &\qquad = (-1)^{\alpha+1} i^\alpha \left[\sum_{s=1}^m (n_s\omega_s)^\alpha\right] F_{n_1 \cdots n_m}(r), \end{aligned} \tag{3.20}$$

where i in (3.20) represents $\sqrt{-1}$. Choose α even and set $M = 1/(\min_{0 \leqq s \leqq m} |\omega_s|^\alpha)$. Then (3.20) implies

$$|F_{n_1 \cdots n_m}(r)| \leqq \frac{MB}{|n_1|^\alpha + \cdots + |n_m|^\alpha}. \tag{3.21}$$

From (3.15), (3.16) and (3.21),

$$|u(\phi, r)| \leqq \left(\frac{MB}{A}\right) \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_m=-\infty}^{\infty} \frac{|n|^{m+1}}{|n_1|^\alpha + \cdots + |n_m|^\alpha}, \qquad |n| \neq 0. \tag{3.22}$$

Hölder's inequality implies

$$|n| \leqq \left(\sum_{s=1}^m |n_s|^\alpha\right)^{1/\alpha} m^{1/\beta}, \tag{3.23}$$

where $\beta=\alpha/(\alpha-1)$. Then (3.22) becomes

$$|u(\phi, r)| \leqq C \sum_{n_1-\infty}^{\infty} \cdots \sum_{n_m=-\infty}^{\infty} \frac{1}{|n|^{\alpha-m-1}}, \qquad |n| \neq 0 \tag{3.24}$$

for some positive constant C. The series on the right of (3.24) converges or diverges with

$$\sum_{|n|=1}^{\infty}\left(\sum_{|n|} \frac{1}{|n|^{\alpha-m-1}}\right) \tag{3.25}$$

(see, e.g., Hyslop [16]). However, there are $2^{m-j}\binom{m}{j}\binom{p-1}{m-j-1}$ integral solutions to $|n_1|+\cdots+|n_m|=|n|=p$, where exactly j, $0 \leqq j \leqq m-1$, of the n_i are 0 (see, e.g., Berman and Fryer [6]). Furthermore $\binom{p-1}{r-1} \leqq p^{r-1}/(r-1)!$ Therefore, (3.25) satisfies

$$\sum_{p=1}^{\infty}\left(\sum_{|n|=p} \frac{1}{p^{\alpha-m-1}}\right) \leqq \sum_{r=1}^{m}\left(\frac{2^r}{(r-1)!}\right)\binom{m}{m-r}\left(\sum_{p-r}^{\infty} \frac{1}{p^{\alpha-m-r}}\right). \tag{3.26}$$

The series on the right of (3.26) converges for each $r=1,2,\cdots,m$ provided $\alpha \geqq 2m+2$ and α even as assumed above.

A similar argument can be used to establish the differentiability of u. To do this, however, note that the higher order derivatives (2.3) can also be written in terms of generalized multilinear forms. If $(a_{11}, a_{21}), \cdots, (a_{1k}, a_{2k}) \in \Sigma$, then by induction on k, using (2.8), if $u \in P_\omega^\alpha(\Sigma)$,

$$D^k u(\phi, r) \cdot (a_{11}, a_{21}) \cdots (a_{1k}, a_{2k}) = \sum_{i_1=1}^{2} \cdots \sum_{i_k=1}^{2} D_{i_1 \cdots i_k} u(\phi, r) \cdot a_{i_1 1} \cdots a_{i_k k}, \tag{3.27}$$

where $D_{i_1 \cdots i_k} u(\phi, r)$, $i_1, \cdots, i_k=1,2$, are the higher order partial derivatives arising from (2.1). For example, if $h \in G^m$, $k \in R^m$, then one has by direct computation

$$D_{12} u(\phi, r) \cdot h \cdot k=\left(\sum_{j_1=1}^{m} \sum_{j_2=1}^{m}\left(\frac{\partial^2 u_i}{\partial r_{j_2}\, \partial \phi_{j_1}}\right) h_{j_2} k_{j_1}\right)_{i=1,\cdots,m}. \tag{3.28}$$

Let p, q be positive integers such that $p+q=k$. From (3.27) one needs only estimate, using (3.15),

$$|D_1^p D_2^q u(\phi, r)| \leqq \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_m=-\infty}^{\infty}\left(\frac{|D^q F_{n_1 \cdots n_m}(r)|}{|(n, \omega)|}\right) \cdot |D^p e^{i(n_1 \omega_1 \phi_1+\cdots+n_m \omega_m \phi_m)}|. \tag{3.29}$$

Since $E \in P_\omega^\alpha(\Sigma)$, in (3.12), then $D^q F_{n_1 \cdots n_m}(r)$ is $\alpha-q$ times continuously differentiable in G^m, $0 \leqq q \leqq \alpha$. Applying integration by parts to $D_2^q u(\phi, r)$ implies that there exists a constant B_1 such that

$$|D^q F_{n_1 \cdots n_m}(r)| \leqq \frac{M B_1}{|n_1|^\alpha+\cdots+|n_m|^\alpha}, \tag{3.30}$$

where α has been assumed even as before. By a straightforward computation, there exists a contant C_1 such that

$$|D^p e^{i(n_1 \omega_1 \phi_1+\cdots+n_m \omega_m \phi_m)}| \leqq C_1 \sum_{j_1+\cdots+j_m=p} |n_1|^{j_1} \cdots |n_m|^{j_m} \leqq W|n|^p \tag{3.31}$$

for some constant $W>0$, since $|n_1|^{j_1} \cdots |n_m|^{j_m} \leqq (\sum_{j=1}^m |n_j|)^p$. Combining (3.16), (3.29), (3.30) and (3.31) gives that there exists some constant $C_2>0$ such that

$$|D_1^p D_2^q u(\phi, r)| \leqq C_2 \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_m=-\infty}^{\infty}\left(\frac{1}{|n|^{\alpha-m-p-1}}\right), \qquad |n| \neq 0. \tag{3.32}$$

The series on the right converges or diverges with

$$\sum_{|n|=1}^{\infty}\left(\sum_{|n|}\frac{1}{|n|^{\alpha-m-p-1}}\right), \tag{3.33}$$

and, as in the case of (3.26),

$$\sum_{s=1}^{\infty}\left(\sum_{|n|=s}\frac{1}{|n|^{\alpha-m-p-1}}\right)\leqq\sum_{r=1}^{m}\left(\frac{2^r}{(r-1)!}\right)\binom{m}{m-r}\left(\sum_{s=r}^{\infty}\frac{1}{s^{-m-p-s}}\right). \tag{3.34}$$

The series on the right converges provided, for $r=1,2,\cdots,m$, $\alpha-p\geqq 2m+2$.

For an Nth order expansion it is clear that $0\leqq p, q\leqq N+1$, so that one must choose α even and $\alpha\geqq N+2m+3\geqq p+2m+2$, which would imply that $\alpha-p\geqq 2m+2$. This proves the lemma. For a similar result in the analytic case see Bogoliubov, Mitropolsky and Samolenko [8].

Equation (3.27) can be extended by repetitive application of the product rule,

$$\begin{aligned} &D^p(D^kf(\phi,r)\cdot(u_{11},u_{21})\cdots(u_{1k},u_{2k}))(s_{11},s_{21})\cdots(s_{1p},s_{2p)}\\ &\quad=\sum_{n_0+\cdots+n_k=p}D^{k+n_0}f(\phi,r)\cdot(D^{n_1}u_{11},D^{n_1}u_{21})\\ &\qquad\cdots(D^{n_k}u_{1k},D^{n_k}u_{2k})\cdot(s_{11},s_{21})\cdots(s_{1p},s_{2p}), \end{aligned} \tag{3.35}$$

where $n_\nu=\sum_{\mu=1}^{p}m_{\nu\mu}$, $\nu=0,1,\cdots,k$ and $\sum_{\nu=1}^{k}m_{\nu\mu}=1$ for $\mu=1,2,\cdots,p$. There are sk^P terms in the sum.

LEMMA 3.2. *Given* $N\geqq 1$, $\Theta, X\in P_\omega^\alpha(\Sigma)$, *where* ω *satisfies* (3.16), $\alpha\geqq N+2m+3$, α *even, then, from* (3.10) *and* (3.19), u_i, w_i, Φ_i, $R_i\in P_\omega^{N-i-1}(\Sigma)$.

Proof. From (3.9) F_1, $G_1\in P_\omega^\alpha(\Sigma)$ and are, therefore, continuously differentiable of order N, and from Lemma 3.1 u_1, w_1 are continuously differentiable of order N. From (3.14), (3.19), Φ_1, R_1 are continuously differentiable of order N in R.

Inductively suppose that Φ_k, R_k, u_k, w_k are continuously differentiable of order $N-k+1$ for $k=1,\cdots,i-1$, and that u_k, w_k are periodic with vector frequency ω. From (2.8), (3.9) and (3.35), one can write

$$\begin{aligned} &D^{N-i+1}F_i(\phi,r)\\ &\quad=\sum_{k=1}^{i=1}\Bigg(\left(\frac{1}{k!}\right)\sum_{j_1+\cdots+j_k=i-1}\ \sum_{n_0+\cdots+n_k=N-i+1}D^{k+n_0}\Theta(\phi,r)\\ &\qquad\cdot(D^{n_1}u_{j_1},D^{n_1}w_{j_1})\cdots(D^{n_k}u_{j_k},D^{n_k}w_{j_k})\\ &\qquad-\sum_{n_0+n_1=N-i+1}D^{n_0+1}u_k(\phi,r)\cdot(D^{N_1}\Phi_{i-k},D^{n_1}R_{i-k})\Bigg), \end{aligned} \tag{3.36}$$

and similarly for $D^{N-i+1}G_i(\phi,r)$. By the inductive hypothesis, Φ_j, R_j are differentiable of order $N-j+1$ for $j=1,\cdots,i-1$. Set $j=i-k$. Then Φ_{i-k}, R_{i-k} are continuously differentiable of order $N-(i-k)+1$ for $i-k=1,\cdots,i-1$ or for $N-i+2$ to N. Therefore, for $0\leqq n_1\leqq N-i+1$, $D^{n_1}\Phi_{i-k}$, $D^{n_1}R_{i-k}$ exist and are continuous. Again by the inductive hypothesis, u_k is differentiable of order $N-k+1$ for $k=1,\cdots,i-1$ and periodic with vector frequency ω in ϕ. For $0\leqq n_0\leqq N-i+1$, one has $1\leqq n_0+1\leqq N-i+2\leqq N-k+1$ as above. Therefore, $D^{n_0}u_k(\phi,r)$ exists continuously and is periodic with vector frequency ω in ϕ. A similar argument holds for w_k. Finally, if $1\leqq k\leqq i-1$ and $0\leqq n_0\leqq N-i+1$, then $1\leqq k+n_0\leqq N$, and therefore, $D^{k+n_0}\Theta(\phi,r)$ exists continuously and is periodic of vector frequency ω in ϕ. Combining these, (3.36) shows that $D^{N-i+1}F_i(\phi,r)$ exists continuously and is periodic with vector frequency ω in ϕ, and similarly for G_i. Now, from (3.15) u_i, w_i are also continuously differentiable of order $N-i+1$ and periodic in ϕ with vector frequency ω by uniform convergence.

4. Main result. In this paper the Nth order approximation to the solution of (3.1), for $N \geqq 1$, will be defined as

$$(4.1) \qquad \theta_N = \phi_N + \sum_{j=1}^{N-1} \varepsilon^j u_j(\phi_N, r_N), \qquad x_N = r_N + \sum_{j=1}^{N-1} \varepsilon^j w_j(\phi_N, r_N),$$

where (ϕ_N, r_N) is the solution to

$$(4.2) \qquad \dot{\phi} = d + \sum_{j=1}^{N} \varepsilon^j \Phi_j(r), \qquad \dot{r} = \sum_{j=1}^{N} \varepsilon^j R_j(r).$$

Φ_j, R_j are defined by (3.9) and (3.19), and u_j, w_j are solutions of (3.10). The initial conditions for (4.2) are given by

$$(4.3) \qquad \phi_N(0, \varepsilon) = \phi_{n0}(\varepsilon), \qquad r_N(0, \varepsilon) = r_{N0}(\varepsilon).$$

These are implicitly defined by the relations

$$(4.4) \qquad \begin{aligned} \phi_{N0}(\varepsilon) &= \theta_0 - \sum_{j=1}^{N-1} \varepsilon^j u_j(\phi_{N0}(\varepsilon), r_{N0}(\varepsilon)) + O(\varepsilon^N), \\ r_{N0}(\varepsilon) &= x_0 - \sum_{j=1}^{N-1} \varepsilon^j w_j(\phi_{N0}(\varepsilon), r_{N0}(\varepsilon)) + O(\varepsilon^n), \end{aligned}$$

where (θ_0, x_0) is the initial condition for (3.1).

For ε sufficiently small, the initial conditions (4.4) can be written explicitly in the form

$$(4.5) \qquad \phi_{N0}(\varepsilon) = \alpha_0 + \sum_{j=1}^{N-1} \varepsilon^j \alpha_j + O(\varepsilon^N), \qquad r_{N0}(\varepsilon) = \beta_0 + \sum_{j=1}^{N-1} \varepsilon^j \beta_j + O(\varepsilon^N).$$

This follows from the implicit function theorem.

In (4.4), expand u_j, w_j in Taylor series. Introduce (4.5) for ϕ_{N0}, r_{N0}, and use (2.10) and (2.9) to give

$$(4.6) \qquad \begin{aligned} &u_j(\phi_{N0}(\varepsilon), r_{N0}(\varepsilon)) \\ &\quad = u_j(\alpha_0, \beta_0) + \sum_{k=1}^{N-1} \sum_{i=k}^{N-1} \varepsilon^i \left[\left(\frac{1}{k!}\right) D^k u_j(\alpha_0, \beta_0) \cdot \sum_{j_1+\cdots+j_k=i} (\alpha_{j_1}, \beta_{j_1}) \cdots (\alpha_{j_k}, \beta_{j_k}) \right] \\ &\qquad + O(\varepsilon^N), \\ &w_j(\phi_{N0}(\varepsilon), r_{N0}(\varepsilon)) \\ &\quad = w_j(\alpha_0, \beta_0) + \sum_{k=1}^{N-1} \sum_{i=k}^{N-1} \left[\left(\frac{1}{k!}\right) D^k w_j(\alpha_0, \beta_0) \cdot \sum_{j_1+\cdots+j_k=i} (\alpha_{j_1}, \beta_{j_1}) \cdots (\alpha_{j_k}, \beta_{j_k}) \right] \\ &\qquad + O(\varepsilon^N) \end{aligned}$$

for $j = 1, \cdots, N-1$. Substitute (4.6) into (4.4) on the right, and use (4.5) on the left and the summation interchange formula

$$\sum_{j=1}^{N-1}\sum_{i=1}^{N-1}\varepsilon^{i+j}A(i,j)=\sum_{j=2}^{N-1}\varepsilon^{j}\left(\sum_{i=1}^{j-1}A(i,j-i)\right)+O(\varepsilon^{N}) \tag{4.7}$$

on the right. Equating powers of ε gives

$$\begin{aligned}
&\alpha_0=\theta_0,\\
&\beta_0=x_0,\\
&\alpha_1=-u_1(\alpha_0,\beta_0),\\
&\beta_1=-w_1(\alpha_0,\beta_0),\\
&\alpha_i=-u_i(\alpha,\beta_0)-\sum_{j=1}^{i-1}\sum_{k=1}^{i-j}\left(\frac{1}{k!}\right)D^k u_j(\alpha_0,\beta_0)\cdot\sum_{j_1+\cdots+j_k=i-j}(\alpha_{j_1},\beta_{j_1})\cdots(\alpha_{j_k},\beta_{j_k}),\\
&\beta_i=-w_i(\alpha_0,\beta_0)-\sum_{j=1}^{i-1}\sum_{k=1}^{i-j}\left(\frac{1}{k!}\right)D^k w_j(\alpha_0,\beta_0)\cdot\sum_{j_1+\cdots+j_k=i-j}(\alpha_{j_1},\beta_{j_1})\cdots(\alpha_{j_k},\beta_{j_k})
\end{aligned} \tag{4.8}$$

for $i=2,3,\cdots,N-1$.

THEOREM 4.1. *Given $L>0$, $N\geqq 1$, G^m a convex region in R^m, and pick $r_0\in G^m$. Let Θ, X in* (3.1) *lie in $P_\omega^\alpha(\Sigma)$, α even, $\alpha\geqq N+2m+3$, where for all $n=(n_1,\cdots,n_m)$, $|n|\neq 0$,*

$$|(n,\omega)|\geqq A|n|^{-(m+1)}, \tag{4.9}$$

$A>0$, constant. Let

$$\frac{dr}{d\tau}=R_1(r),\qquad r(0)=r_0 \tag{4.10}$$

have a solution $r=g_1(\tau)$ which remains in G^m for $0\leqq\tau\leqq L$. Then:

(1) *There exists an $\varepsilon_N>0$ such that for $0<\varepsilon\leqq\varepsilon_N$ the system*

$$\begin{aligned}
\dot\theta&=d+\varepsilon\Theta(\theta,x),\qquad \theta(0,\varepsilon)=\theta_0,\\
\dot x&=\varepsilon X(\theta,x),\qquad x(0,\varepsilon)=x_0
\end{aligned} \tag{4.11}$$

has a unique solution $(\theta(t,\varepsilon),x(t,\varepsilon))$, which remains in $R^m\times G^m$ for $0\leqq t\leqq L/\varepsilon$.

(2) *The autonomous system*

$$\begin{aligned}
\dot\phi&=d+\sum_{j=1}^{N}\varepsilon^j\Phi_j(r),\qquad \phi_N(0,\varepsilon)=\phi_{N0}(\varepsilon),\\
\dot r&=\sum_{j=1}^{N}\varepsilon^j R_j(r),\qquad r_N(0,\varepsilon)=r_{N0}(\varepsilon),
\end{aligned} \tag{4.12}$$

with Φ_j, R_j defined by (3.9) *and* (3.19), *ϕ_{N0}, r_{N0} defined by* (4.4), *has a unique solution $(\phi_N(t,\varepsilon),r_N(t,\varepsilon))$, which remains in $R^m\times G^m$ for $0\leqq t\leqq L/\varepsilon$.*

(3) *The* Nth *order approximation is given by*

$$\begin{aligned}
\theta_N(t,\varepsilon)&=\phi_N(t,\varepsilon)+\sum_{j=1}^{N-1}\varepsilon^j u_j(\phi_N(t,\varepsilon),r_N(t,\varepsilon)),\\
x_N(t,\varepsilon)&=r_N(t,\varepsilon)+\sum_{j=1}^{N-1}\varepsilon^j w_j(\phi_N(t,\varepsilon),r_N(t,\varepsilon)),
\end{aligned} \tag{4.13}$$

where u_j, w_j *are solutions of* (3.10), *periodic in* ϕ *with angular vector frequency* ω, *and continuously differentiable of order* $N-j+1$. *The* Nth *order approximation satisfies*

$$|\theta(t,\varepsilon)-\theta_N(t,\varepsilon)|\leqq C_N\varepsilon^N,\qquad |x(t,\varepsilon)-x_N(t,\varepsilon)|\leqq C_N\varepsilon^N, \tag{4.14}$$

for $0\leqq t\leqq L/\varepsilon$, C_N *some positive constant dependent on* N *and independent of* ε.

Proof. The proof begins with showing that the autonomous system (4.12) has a unique solution $(\phi_N(t,\varepsilon), r_N(t,\varepsilon))$ that remains in some set $R^m\times S_1^m$, where $S_1^m\subset G^m$ is convex and compact, for $0\leqq t\leqq L/\varepsilon$ for $0<\varepsilon\leqq\varepsilon_N$, some $\varepsilon_N>0$.

By hypothesis, $r=g_1(\tau)$ remains in G^m for $0\leqq\tau\leqq L$. Let

$$S_0^m=\{r\in R^m: r=g_1(\tau), 0\leqq\tau\leqq L\}. \tag{4.15}$$

As a continuous image of a compact set S_0^m is compact. Since G^m is an open domain and $S_0^m\subset G^m$, there exists $\rho_0>0$ such that

$$\rho_0=\inf_{\substack{x\in S_0^m\\ y\in R^m-G^m}}|x-y|. \tag{4.16}$$

Let $w_N(\tau,\varepsilon)$ be the solution of

$$\frac{dw}{d\tau}=R_1(w)+\cdots+\varepsilon^{N-1}R_N(w),\qquad w(0,\varepsilon)=r_{N0}(\varepsilon), \tag{4.17}$$

where $r_{N0}(\varepsilon)$ is defined by (4.5), (4.8). For $w\in G^m$, $R_j(w)$ is continuously differentiable for $j=1,2,\cdots,N$ as shown previously. Then from standard theorems on existence, uniqueness of solutions and continuity with respect to parameters and initial conditions there exists an $\varepsilon_N>0$ and a unique solution $w_N(\tau,\varepsilon)$ of (4.17), continuous with respect to ε such that for $0\leqq\tau\leqq L$, $0<\varepsilon\leqq\varepsilon_N$,

$$|w_N(\tau,\varepsilon)-g_1(\tau)|<\rho_0. \tag{4.18}$$

The fact that $|r_{N0}(\varepsilon)-R_0|\leqq C_N\varepsilon$ from (4.5) for some C_N, $N\geqq 1$, has been used to establish (4.18).

Now define the set

$$S_1=\{w\in R^m: w=w_N(\tau,\varepsilon), 0\leqq\tau\leqq L, 0\leqq\varepsilon\leqq\varepsilon_N\}. \tag{4.19}$$

$S_1\subset G^m$ and is compact. Let $H(S_1)$ be the convex hull of S_1. $H(S_1)$ is compact (Stoer and Witzgall [32]). Since G^m is convex, $H(S_1)\subset G^m$.

Let

$$\rho_1=\inf_{\substack{x\in H(S_1)\\ y\in R^m-G^m}}|x-y|. \tag{4.20}$$

Since $H(S_1)$ is compact and in G^m, which is open, $\rho_1>0$.

By uniqueness the solution of

$$\frac{dr}{dt}=\varepsilon R_1(r)+\cdots+\varepsilon^N R_N(r),\qquad r(0,\varepsilon)=r_{N0}(\varepsilon) \tag{4.21}$$

is given by

$$r_N(t,\varepsilon)=w_N(\varepsilon t,\varepsilon). \tag{4.22}$$

Therefore,

$$\{r \in R^m : r = r_N(t, \varepsilon), 0 \leqq t \leqq L/\varepsilon, 0 < \varepsilon \leqq \varepsilon_N\} \subset H(S_1) \subset G^m. \tag{4.23}$$

If $r_N(t, \varepsilon)$ is inserted into (4.12), let $\phi_N(t, \varepsilon)$ be the solution of the ϕ equation subject to $\phi_N(0, \varepsilon) = \phi_{N0}(\varepsilon)$. Define the improved Nth order approximation by

$$\begin{aligned} \tilde{\theta}_N(t, \varepsilon) &= \theta_N(t, \varepsilon) + \varepsilon^N u_N(\phi_N(t, \varepsilon), r_N(t, \varepsilon)), \\ \tilde{x}_N(t, \varepsilon) &= x_N(t, \varepsilon) + \varepsilon^N w_N(\phi_N(t, \varepsilon), r_N(t, \varepsilon)), \end{aligned} \tag{4.24}$$

where θ_N, x_N are constructed in (4.13).

Let $(\theta(t, \varepsilon), x(t, \varepsilon))$ represent the unique solution of (4.11) on its maximal interval of solution $0 \leqq t \leqq t_1(\varepsilon)$.

Define

$$S_2 = \{y \in R^m : |y - w| \leqq \rho_1/2 \text{ for some } w \in H(S_1)\}. \tag{4.25}$$

S_2 is closed and bounded and therefore compact. S_2 is also convex. For, let $y_1, \cdots, y_p \in S_2$, $\alpha_i > 0$, $\sum_{i=1}^p \alpha_i = 1$. Consider $\alpha_1 y_1 + \cdots + \alpha_p y_p$. To each y_i there corresponds $w_i \in H(S_1)$ satisfying $|y_i - w_i| \leqq \rho_1/2$. But by convexity $\alpha_1 w_1 + \cdots + \alpha_p w_p \in H(S_1)$, and therefore $|(\alpha_1 y_1 + \cdots + \alpha_p y_p) - (\alpha_1 w_1 + \cdots + \alpha_p w_p)| \leqq \rho_1/2$. This means that

$$\operatorname{dist}(S_2, R^m - G^m) > \frac{\rho_1}{2}. \tag{4.26}$$

From (4.23), $r_N(t, \varepsilon) \in H(S_1)$ for $0 \leqq t \leqq L/\varepsilon$, $0 < \varepsilon \leqq \varepsilon_N$. Furthermore, u_j, w_j are continuously differentiable of order $N - j + 1$ and periodic in ϕ. They are bounded on $R^m \times H(S_1)$, since $H(S_1)$ is compact. Therefore, from (4.25), (4.24) and (4.13), there exists an $\varepsilon_N > 0$ such that $\tilde{x}_N(t, \varepsilon) \in S_2$ or $(\tilde{\theta}_N(t, \varepsilon), \tilde{x}_N(t, \varepsilon)) \in R^m \times S_2$. By the convexity of S_2

$$\phi_N(t, \varepsilon) + \lambda[\tilde{\theta}_N(t, \varepsilon) - \phi_N(t, \varepsilon)], \qquad r_N(t, \varepsilon) + \lambda[\tilde{x}_N(t, \varepsilon) - r_N(t, \varepsilon)] \tag{4.27}$$

are in $R^m \times S_2$ for $0 \leqq t \leqq L/\varepsilon$, $0 < \varepsilon \leqq \varepsilon_N$ and $0 \leqq \lambda \leqq 1$.

From (4.24), (4.13), (4.12), (2.8), one can write

$$\begin{aligned} \dot{\theta} - \dot{\tilde{\theta}}_N &= \varepsilon[\Theta(\theta, x) - \Theta(\tilde{\theta}_N, \tilde{x}_N)] + \varepsilon\Theta(\tilde{\theta}_N, \tilde{x}_N) \\ &\quad - \sum_{j=1}^{N} \varepsilon^j F_j(\phi_N, r_N) - \sum_{j=1}^{N}\sum_{i=1}^{N} \varepsilon^{j+i} Du_j(\phi_N, r_N) \cdot (\Phi_i(r_N), R_i(r_N N)), \\ \dot{x} - \dot{\tilde{x}}_N &= \varepsilon[X(\theta, x) - X(\tilde{\theta}_N, \tilde{x}_N)] + \varepsilon X(\tilde{\theta}_N, \tilde{x}_N) \\ &\quad - \sum_{j=1}^{N} \varepsilon^j G_j(\phi_N, r_N) - \sum_{j=1}^{N}\sum_{i=1}^{N} \varepsilon^{j+i} Dw_j(\phi_N, r_N) \cdot (\Phi_i(r_n), R_i(r_N)). \end{aligned} \tag{4.28}$$

Using (2.6), write

$$\begin{aligned} \Theta(\tilde{\theta}_N, \tilde{x}_N) &= \sum_{k=0}^{N-1} \left(\frac{1}{k!}\right) D^k \Theta(\phi_N, r_N) \cdot (\tilde{\theta}_N - \phi_N, \tilde{x}_N - r_N)^k + E_{1N}(\phi_N, r_N, \varepsilon), \\ X(\tilde{\theta}_N, \tilde{x}_N) &= \sum_{k=0}^{N-1} \left(\frac{1}{k!}\right) D^k X(\phi_N, r_N) \cdot (\tilde{\theta}_N - \phi_N, \tilde{x}_N - r_N)^k + E_{2N}(\phi_N, r_N, \varepsilon) \end{aligned} \tag{4.29}$$

where

$$E_{1N}(\phi_N, r_N, \varepsilon)$$
$$= \left[\left(\frac{1}{(N-1)!}\right)\int_0^1 (1-\gamma)^{N-1} D^N\Theta(\phi_N+\gamma(\tilde{\theta}_N-\phi_N), r_N+\gamma(\tilde{x}_N-r_N))\,d\gamma\right]$$
$$\cdot(\tilde{\theta}_N-\phi_N, \tilde{x}_N-r_N)^N$$

(4.30)

$$E_{2N}(\phi_N, r_N, \varepsilon)$$
$$= \left[\left(\frac{1}{(N-1)!}\right)\int_0^1 (1-\gamma)^{N-1} D^N X(\phi_N+\gamma(\tilde{\theta}_N-\phi_N), r_N+\gamma(\tilde{x}_N-r_N))\,d\gamma\right]$$
$$\cdot(\tilde{\theta}_N-\phi_N, \tilde{x}_N-r_N)^N.$$

Using (4.13) and (4.24), define h_1, h_2 so that

$$\varepsilon h_1 = \tilde{\theta}_N-\phi_N = \varepsilon\sum_{j=1}^{N}\varepsilon^{j-1}u_j(\phi_N, r_N),$$
$$\varepsilon h_2 = \tilde{x}_N - r_N = \varepsilon\sum_{j=1}^{N}\varepsilon^{j-1}w_j(\phi_N, r_N). \tag{4.31}$$

From (4.29), (2.9) and changing indices by setting $j=i+1$, one can write

$$\Theta(\tilde{\theta}_N, \tilde{x}_N) = \sum_{k=0}^{N-1}\left(\frac{1}{k!}\right)D^k\Theta(\phi_N, r_N)$$
$$\cdot\sum_{j=k+1}^{kN+1}\varepsilon^{j-1}\sum_{j_1+\cdots+j_k=j-1}(u_{j_1}, w_{j_1})\cdots(u_{j_k}, w_{j_k})+\varepsilon^N\bar{E}_{1n}(\phi_N, r_N, \varepsilon),$$

(4.32)

$$X(\tilde{\theta}_N, \tilde{x}_N) = \sum_{k=0}^{N-1}\left(\frac{1}{k!}\right)D^kX(\phi_N, r_N)$$
$$\cdot\sum_{j=k+1}^{kN+1}\varepsilon^{j-1}\sum_{j_1+\cdots+j_k=j-1}(u_{j_1}, w_{j_1})\cdots(u_{j_k}, w_{j_k})+\varepsilon^N\bar{E}_{2N}(\phi_N, r_N, \varepsilon),$$

where

$$\bar{E}_{1N}(\phi_N, r_N, \varepsilon)$$
$$= \left[\left(\frac{1}{(N-1)!}\right)\int_0^1 (1-\gamma)^{N-1}D^N\Theta(\phi_N+\varepsilon\gamma h_1, r_N+\varepsilon\gamma h_2)\,d\gamma\right]\cdot(h_1, h_2)^N, \tag{4.33}$$
$$\bar{E}_{2n}(\phi_N, r_N, \varepsilon)$$
$$= \left[\left(\frac{1}{(N-1)!}\right)\int_0^1 (1-\gamma)^{N-1}D^NX(\phi_N+\varepsilon\gamma h_1, r_N+\varepsilon\gamma h_2)\,d\gamma\right]\cdot(h_1, h_2)^N.$$

We make use of the following algebraic relation in (4.32):

$$\sum_{k=0}^{N-1}\sum_{j=k+1}^{kN+1}\varepsilon^{j-1}A(k,j) = A(0,1)+\sum_{j=2}^{N}\varepsilon^{j-1}\sum_{k=1}^{j-1}A(k,j)+\sum_{k=1}^{N-1}\sum_{j=N+1}^{kN+1}\varepsilon^{j-1}A(k,j). \tag{4.34}$$

If we accumulate all terms that include the Nth or higher power into a new error term for each equation, we can write (4.32) as

$$\Theta(\tilde{\theta}_N, \tilde{x}_N) = \Theta(\phi_N, r_N) + \sum_{j=2}^{N} \varepsilon^{j-1} \sum_{k=1}^{j-1} \left(\frac{1}{k!}\right) D^k \Theta(\phi_N, r_N) \cdot \sum_{j_1+\cdots+j_k=j-1} (u_{j_1}, w_{j_1}) \cdots (u_{j_k}, w_{j_k}) + \varepsilon^N \bar{\bar{E}}_{1N}(\phi_N, r_N, \varepsilon),$$

(4.35)

$$X(\tilde{\theta}_N, \tilde{x}_N) = X(\phi_N, r_N) + \sum_{j=2}^{N} \varepsilon^{j-1} \sum_{k=1}^{j-1} \left(\frac{1}{k!}\right) D^k X(\phi_N, r_N) \cdot \sum_{j_1+\cdots+j_k=j-1} (u_{j_1}, w_{j_1}) \cdots (u_{j_k}, w_{j_k}) + \varepsilon^N \bar{\bar{E}}_{2N}(\phi_N, r_N, \varepsilon).$$

Introduce (3.9) into (4.35) next; then substitute (4.35) into (4.28) and get, noting the definitions of F_1, G_1 in (3.9),

$$\dot{\theta} - \dot{\tilde{\theta}}_N = \varepsilon[\Theta(\theta, x) - \Theta(\tilde{\theta}_N, \tilde{x}_N)] + \sum_{j=2}^{N} \varepsilon^j \left[\sum_{k=1}^{j-1} Du_k(\phi_N, r_N) \cdot (\Phi_{j-k}, R_{j-k})\right] - \sum_{j=1}^{N} \sum_{i=1}^{N} \varepsilon^{j+i} Du_j(\phi_N, r_N) \cdot (\Phi_i, R_i) + \varepsilon^{N+1} \bar{\bar{E}}_{1N}(\phi_N, \tau_N, \varepsilon),$$

(4.36)

$$\dot{x} - \dot{\tilde{x}}_N = \varepsilon[X(\theta, x) - X(\tilde{\theta}_N, \tilde{x}_N)] + \sum_{j=2}^{N} \varepsilon^j \left[\sum_{k=1}^{j-1} Dw_k(\phi_N, r_N) \cdot (\Phi_{j-k}, R_{j-k})\right] - \sum_{j=1}^{N} \sum_{i=1}^{N} \varepsilon^{j+i} Dw_j(\phi_N, r_N) \cdot (\Phi_i, R_i) + \varepsilon^{N+1} \bar{\bar{E}}_{2N}(\phi_N, r_N, \varepsilon).$$

Now make use of the relation

(4.37) $$\sum_{j=1}^{N} \sum_{i=1}^{N} \varepsilon^{j+i} A(j, i) = \sum_{j=2}^{N} \varepsilon^j \sum_{k=1}^{j-1} A(k, j-k) + \varepsilon^N \sum_{j=1}^{N} \varepsilon^j \sum_{k=j}^{N} A(k, N+j-k),$$

and rewrite (4.36) as

$$\dot{\theta} - \dot{\tilde{\theta}}_N = \varepsilon[\Theta(\theta, x) - \Theta(\tilde{\theta}_N, \tilde{x}_N)] + \varepsilon^{N+1}\left[\bar{\bar{E}}_{1N}(\phi_N, r_N, \varepsilon) - \sum_{j=1}^{N} \varepsilon^{j-1} \sum_{k=j}^{N} Du_k(\phi_N, r_N) \cdot (\Phi_{N+j-k}, R_{N+j-k})\right],$$

(4.38)

$$\dot{x} - \dot{\tilde{x}}_N = \varepsilon[X(\theta, x) - X(\tilde{\theta}_N, \tilde{x}_N)] + \varepsilon^{N+1}\left[\bar{\bar{E}}^2{}_N(\phi_N, r_N, \varepsilon) - \sum_{j=1}^{N} \varepsilon^{j-1} \sum_{k=j}^{N} Dw_k(\phi_N, r_N) \cdot (\Phi_{N+j-k}, R_{N+j-k})\right].$$

Since $\Theta, X \in P_\omega^\alpha(\Sigma)$ for α even and $\alpha \leqq N+2m+3$ and $(\phi_N + \varepsilon\gamma h_1, r_N + \varepsilon\gamma h_2) \in R^m + S_2$ for $0 \leqq t \leqq L/\varepsilon$, $0 < \varepsilon \leqq \varepsilon_N$, $0 \leqq \gamma \leqq 1$, because of (4.31) and (4.27), and since S_2 is compact there exists a constant, call it C_N again, such that

(4.39) $$|\bar{\bar{E}}_{1N}(\phi_N, r_N, \varepsilon)| \leqq C_N, \qquad |\bar{\bar{E}}_{2N}(\phi_N, r_N, \varepsilon)| \leqq C_N.$$

Since u_i, w_i are continuously differentiable and $(\phi_N, r_N) \in R^m \times S_2$, then for $1 \leqq j \leqq N$,

$j \leqq k \leqq N$, $0 \leqq t \leqq L/\varepsilon$, $0 \leqq \varepsilon \leqq \varepsilon_N$,

$$\begin{aligned} &|Du_k(\phi_N, r_N) \cdot (\Phi_{N+j-k}(r_N), R_{N+j-k}(r_N))| \leqq C_N, \\ &|Dw_k(\phi_N, r_N) \cdot (\Phi_{N+j-k}(r_N), R_{N+j-k}(r_N))| \leqq C_N, \end{aligned} \tag{4.40}$$

where C_N is some constant. Combining the initial conditions for (4.12) with (4.4), (4.1) implies that

$$\theta_N(0, \varepsilon) = \theta_0 + O(\varepsilon^N), \qquad x_N(0, \varepsilon) = x_0 + O(\varepsilon^N). \tag{4.41}$$

Then, at $t = 0$, insert (4.41) into (4.24) and get the inequalities

$$|\theta(0, \varepsilon) - \tilde{\theta}_N(0, \varepsilon)| \leqq C_N \varepsilon^N, \qquad |x(0, \varepsilon) - \tilde{x}_N(0, \varepsilon)| \leqq C_N \varepsilon^N, \tag{4.42}$$

since $\theta(0, \varepsilon) = \theta_0$, $x(0, \varepsilon) = x_0$.

Combining (4.42), (4.40), (4.39) and (4.38) there exists a constant C_N such that for $0 \leqq t \leqq \min(t_1(\varepsilon), L/\varepsilon)$ and $0 < \varepsilon \leqq \varepsilon_N$,

$$\begin{aligned} &|\theta(t, \varepsilon) - \tilde{\theta}_N(t, \varepsilon)| \leqq C_N \varepsilon^N + \varepsilon \int_0^t |\Theta(\theta(s, \varepsilon), x(s, \varepsilon)) - \Theta(\tilde{\theta}_N(s, \varepsilon), \tilde{x}_N(s, \varepsilon))| ds, \\ &|x(t, \varepsilon) - \tilde{x}_N(t, \varepsilon)| \leqq C_N \varepsilon^N + \varepsilon \int_0^t |X(\theta(s, \varepsilon), x(s, \varepsilon)) - X(\tilde{\theta}_N(s, \varepsilon), \tilde{x}_N(s, \varepsilon))| ds. \end{aligned} \tag{4.43}$$

Now define a new set

$$S_3 = \{y \in R^m : |y - w| \leqq \rho_1/3 \text{ for some } w \in S_2\}. \tag{4.44}$$

Since S_2 is convex and compact, so is S_3. Furthermore $S_3 \subset G^m$.

Suppose there is a first point $t_0(\varepsilon) > 0$ in the open interval $0 < t_0 < L/\varepsilon$ such that $x(t_0(\varepsilon), \varepsilon)$ is a point in the boundary of S_3. Since $t_1(\varepsilon)$ represents the maximal interval of existence then $t_0(\varepsilon) < t_1(\varepsilon)$. Since $\tilde{x}_N(t_0, \varepsilon) \in S_2$ then either $|x(t_0, \varepsilon) - \tilde{x}_N(t_0, \varepsilon)| = \rho_1/3$ or $|x(t_0, \varepsilon) - \tilde{x}_N(t_0, \varepsilon)| > \rho_1/3$, otherwise $x(t_0, \varepsilon)$ would be an interior point of S_3 for $0 < \varepsilon \leqq \varepsilon_N$.

Now Θ, $X \in P_\omega^\alpha(\Sigma)$ for $\alpha \leqq N + 2m + 3$, and S_3 is compact, so by the mean value property there is some constant C_N such that

$$\begin{aligned} &|\Theta(\theta(t, \varepsilon), x(t, \varepsilon)) - \Theta(\tilde{\theta}_N(t, \varepsilon), \tilde{x}_N(t, \varepsilon))| \\ &\qquad \leqq C_N(|\theta(t, \varepsilon) - \tilde{\theta}_N(t, \varepsilon)| + |x(t, \varepsilon) - \tilde{x}_N(t, \varepsilon)|), \\ &|X(\theta(t, \varepsilon), x(t, \varepsilon)) - X(\tilde{\theta}_N(t, \varepsilon), \tilde{x}_N(t, \varepsilon))| \\ &\qquad \leqq C_N(|\theta(t, \varepsilon) - \tilde{\theta}_N(t, \varepsilon)| + |x(t, \varepsilon) - \tilde{x}_N(t, \varepsilon)|) \end{aligned} \tag{4.45}$$

for $0 \leqq t \leqq t_0(\varepsilon)$, $0 < \varepsilon \leqq \varepsilon_N$. Substitute (4.45) into (4.43), and add the two inequalities. This gives

$$\begin{aligned} &|\theta(t, \varepsilon) - \tilde{\theta}_N(t, \varepsilon)| + |x(t, \varepsilon) - \tilde{x}_N(t, \varepsilon)| \\ &\qquad \leqq 2C_N \varepsilon^N + 2C_N \varepsilon \int_0^t [|\theta(s, \varepsilon) - \tilde{\theta}_N(s, \varepsilon)| + |x(s, \varepsilon) - \tilde{x}_N(s, \varepsilon)|] \, ds. \end{aligned} \tag{4.46}$$

By Gronwall's inequality for $0 \leqq t \leqq t_0(\varepsilon) < L/\varepsilon$,

$$|\theta(t, \varepsilon) - \tilde{\theta}_N(t, \varepsilon)| + |x(t, \varepsilon) - \tilde{x}_N(t, \varepsilon)| \leqq C_N \varepsilon^N \tag{4.47}$$

for some C_N, $0 \leqq t \leqq t_0(\varepsilon)$, $0 < \varepsilon \leqq \varepsilon_N$. Therefore,

$$|\theta(t, \varepsilon) - \tilde{\theta}_N(t, \varepsilon)| \leqq C_N \varepsilon^N, \qquad |x(t, \varepsilon) - \tilde{x}_N(t, \varepsilon)| \leqq C_N \varepsilon^N. \tag{4.48}$$

But at $t = t_0(\varepsilon)$ this implies that

$$0<\frac{\rho_1}{3}\leqq |x(t_0,\varepsilon)-\tilde{x}_N(t_0,\varepsilon)|\leqq C_N\varepsilon^N, \tag{4.49}$$

which is impossible for ε sufficiently small. The inequality for θ clearly remains true for $0\leqq t\leqq L/\varepsilon$.

Therefore, there is a unique solution of (4.11) that remains in $R^m\times S_3\subset R^m\times G^m$ and satisfies (4.48) for $0\leqq t\leqq L/\varepsilon$. But from (4.24), since u_N, w_N are bounded on $R^m\times S_2$ by some $C_N>0$, we have

$$\begin{aligned}|\theta-\theta_N|&\leqq|\theta-\tilde{\theta}_N|+|\tilde{\theta}_N-\theta_N|\leqq 2C_N\varepsilon^N,\\ |x-x_N|&\leqq|x-\tilde{x}_N|+|\tilde{x}_N-x_N|\leqq 2C_N\varepsilon^N\end{aligned} \tag{4.50}$$

for $0\leqq t\leqq L/\varepsilon$, $0<\varepsilon\leqq\varepsilon_N$, ε_N sufficiently small. This proves the main result.

5. A coupled van der Pol oscillator. In this section we will apply the averaging algorithm developed in § 3 to compute the second order asymptotic solution to the van der Pol system

$$\ddot{z}_1+\mu_1^2z_1=\varepsilon(1-z_1^2-az_2^2)\dot{z}_1,\qquad \ddot{z}_2+\mu_2^2z_2=\varepsilon(1-\alpha z_1^2-z_2^2)\dot{z}_2, \tag{5.1}$$

where $\varepsilon>0$, $a>0$, $\alpha>0$, and $\mu_1,\mu_2>0$ and satisfy $m_1\mu_1+m_2\mu_2\neq 0$ for m_1, m_2 integers. This system has been studied previously by Hale [15] and Gilsinn [14].

To put (5.1) into form (3.1), first transform it by the variables

$$u_1=z_1,\quad u_2=\dot{z}_1,\quad w_1=z_2,\quad w_2=\dot{z}_2. \tag{5.2}$$

Then (5.1) becomes

$$\begin{aligned}\dot{u}_1&=u_2,&\quad \dot{u}_2&=-\mu_1^2u_1+\varepsilon(1-u_1^2-aw_1^2)u_2,\\ \dot{w}_1&=w_2,&\quad \dot{w}_2&=-\mu_2^2w_1+\varepsilon(1-\alpha u_1^2-w_1^2)w_2.\end{aligned} \tag{5.3}$$

Then introduce, using (1.8) with $\beta=\frac{1}{2}$,

$$\begin{aligned}u_1&=\sqrt{x_1}\sin\mu_1\theta_1,&\quad u_2&=\mu_1\sqrt{x_1}\cos\mu_1\theta_1,\\ w_1&=\sqrt{x_2}\sin\mu_2\theta_2,&\quad w_2&=\mu_2\sqrt{x_2}\cos\mu_2\theta_2,\end{aligned} \tag{5.4}$$

into (5.2), where $x_1, x_2\geqq 0$. Then (5.3) becomes

$$\dot{\theta}=d+\varepsilon\Theta(\theta,x),\qquad \dot{x}=\varepsilon X(\theta,x), \tag{5.5}$$

where

$$d=\begin{pmatrix}1\\1\end{pmatrix},\quad \theta=\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix},\quad x=\begin{pmatrix}x_1\\x_2\end{pmatrix},\quad \Theta=\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix},\quad X=\begin{pmatrix}X_1\\X_2\end{pmatrix}, \tag{5.6}$$

and

$$\begin{aligned}\Theta_1(\theta,x)&=-\left(\frac{1}{2\mu_1}\right)(\sin 2\mu_1\theta_1-2x_1\sin^3\mu_1\theta_1\cos\mu_1\theta_1-ax_2\sin 2\mu_1\theta_1\sin^2\mu_2\theta_2),\\ \Theta_2(\theta,x)&=-\left(\frac{1}{2\mu_2}\right)(\sin 2\mu_y\theta_2-\alpha x_1\sin^2\mu_1\theta_1\sin 2\mu_2\theta_2-2x_2\sin^3\mu_2\theta_2\cos\mu_2\theta_2),\\ X_1(\theta,x)&=2x_1\left(\cos^2\mu_1\theta_1-\left(\frac{x_1}{4}\right)\sin^2 2\mu_1\theta_1-ax_2\cos^2\mu_1\theta_1\sin^2\mu_2\theta_2\right),\\ X_2(\theta,x)&=2x_2\left(\cos^2\mu_2\theta_2-\alpha x_1\sin^2\mu_1\theta_1\cos^2\mu_2\theta_2-\left(\frac{x_2}{4}\right)\sin^2 2\mu_2\theta_2\right).\end{aligned} \tag{5.7}$$

The next step is to compute u_1, u_2, w_1, w_2, Φ_1, Φ_2, R_1, R_2 in (3.2) and (3.3) in order to reduce (5.5) to a form (3.3) with $N=2$. From (3.19), (3.9) and (5.7) we can compute directly that

$$\Phi_1(r)=\begin{pmatrix}0\\0\end{pmatrix},\qquad R_1(r)=\begin{pmatrix} r_1-\left(\frac{r_1^2}{4}\right)-\left(\frac{ar_1r_2}{2}\right)\\ r_2-\left(\frac{\alpha r_1r_2}{2}\right)-\left(\frac{r_2^2}{4}\right)\end{pmatrix}. \tag{5.8}$$

Following (3.10), the next step is to solve for u_1, w_1 so that

$$\begin{aligned} D_1u_1\cdot d&=F_1(\phi,r)-\Phi_1(r)=\Theta(\phi,r),\\ D_1w_1\cdot d&=G_1(\phi,r)-R_1(r)=X(\phi,r)-R_1(r),\end{aligned} \tag{5.9}$$

where

$$\begin{gathered} u_1=\begin{pmatrix}u_{11}\\u_{12}\end{pmatrix},\qquad w_1=\begin{pmatrix}w_{11}\\w_{12}\end{pmatrix},\\ D_1u_1=\begin{pmatrix}\partial u_{11}/\partial\phi_1 & \partial u_{11}/\partial\phi_2\\ \partial u_{12}/\partial\phi_1 & \partial u_{12}/\partial\phi_2\end{pmatrix},\qquad D_1w_1=\begin{pmatrix}\partial w_{11}/\partial\phi_1 & \partial w_{11}/\partial\phi_2\\ \partial w_{12}/\partial\phi_1 & \partial w_{12}/\partial\phi_2\end{pmatrix}.\end{gathered} \tag{5.10}$$

From (5.7), (5.9) and (5.10) one can compute the solutions u_1, w_1 as

$$\begin{aligned} u_{11}&=\left[\frac{1}{4\mu_1^2}-\frac{r_1}{8\mu_1^2}-\frac{ar_2}{8\mu_1^2}\right]\cos 2\mu_1\phi_1+\left(\frac{r_1}{32\mu_1^2}\right)\cos 4\mu_1\phi_1\\ &\quad+\left[\frac{-ar_2}{8(\mu_2^2-\mu_1^2)}\right]\cos 2\mu_1\phi_1\cos 2\mu_2\phi_2+\left[\frac{-a\mu_2r_2}{8\mu_1(\mu_2^2-\mu_1^2)}\right]\sin 2\mu_1\phi_1\sin 2\mu_2\phi_2,\\ u_{12}&=\left[\frac{1}{4\mu_2^2}-\frac{\alpha r_1}{8\mu_2^2}-\frac{r_2}{8\mu_2^2}\right]\cos 2\mu_2\phi_2+\left(\frac{r_2}{32\mu_2^2}\right)\cos 4\mu_2\phi_2\\ &\quad+\left[\frac{-\alpha r_1}{8(\mu_1^2-\mu_2^2)}\right]\cos 2\mu_1\phi_1\cos 2\mu_2\phi_2+\left[\frac{-\alpha\mu_1r_1}{8\mu_2(\mu_1^2-\mu_2^2)}\right]\sin 2\mu_1\phi_1\sin 2\mu_2\phi_2,\\ w_{11}&=\left[\frac{r_1}{2\mu_1}-\frac{ar_1r_2}{4\mu_1}\right]\sin 2\mu_1\phi_1+\left(\frac{r_1^2}{16\mu_1}\right)\sin 4\mu_1\phi_1+\left(\frac{\alpha r_1r_2}{4\mu_2}\right)\sin 2\mu_2\phi_2\\ &\quad+\left[\frac{-a\mu_2r_1r_2}{4(\mu_1^2-\mu_2^2)}\right]\cos 2\mu_1\phi_1\sin 2\mu_2\phi_2+\left[\frac{a\mu_1r_1r_2}{4(\mu_1^2-\mu_2^2)}\right]\sin 2\mu_1\phi_1\cos 2\mu_2\phi_2,\\ w_{12}&=\left[\frac{r_2}{2\mu_2}-\frac{\alpha r_1r_2}{4\mu_2}\right]\sin 2\mu_2\phi_2+\left(\frac{r_2^2}{16\mu_2}\right)\sin 4\mu_2\phi_2+\left(\frac{\alpha r_1r_2}{4\mu_2}\right)\sin 2\mu_1\phi_1\\ &\quad+\left[\frac{-\alpha\mu_2r_1r_2}{4(\mu_1^2-\mu_2^2)}\right]\cos 2\mu_1\phi_1\sin 2\mu_2\phi_2+\left[\frac{\alpha\mu_1r_1r_2}{4(\mu_1^2-\mu_2^2)}\right]\sin 2\mu_1\phi_1\cos 2\mu_2\phi_2.\end{aligned} \tag{5.11}$$

Now from (3.9) solve for $F_2(\phi,r)$, $G_2(\phi,r)$, given by

$$\begin{aligned} F_2(\phi,r)&=D\Theta(\phi,r)(u_1,w_1)-Du_1(\phi,r)(\Phi_1,R_1),\\ G_2(\phi,r)&=DX(\phi,r)(u_1,w_1)-Dw_1(\phi,r)(\Phi_1,R_1)\end{aligned} \tag{5.12}$$

and then compute the means $M_\phi F_2$ and $M_\phi G_2$ from (3.14) and (3.19) and (5.12). Then

$$(5.13)\quad \begin{aligned}
\Phi_{21} &= M_\phi F_{21} = \left(\frac{1}{8\mu_1^2}\right)\left[-1+\frac{3}{2}r_1+ar_2+\left(\frac{-3a(\mu_1^2-\mu_2^2)-a\alpha\mu_2^2}{4(\mu_1^2-\mu_2^2)}\right)r_1r_2\right.\\
&\qquad\left.+\left(\frac{-11}{32}\right)r_1^2+\left(\frac{3a^2\mu_1^2-2a^2\mu_2^2}{8(\mu_2^2-\mu_1^2)}\right)r_2^2\right],\\
\Phi_{22} &= M_\phi F_{22} = \left(\frac{1}{8\mu_2^2}\right)\left[-1+\alpha r_1+\frac{3}{2}r_2+\left(\frac{-3\alpha(\mu_2^2-\mu_1^2)-a\alpha\mu_1^2}{4(\mu_2^2-\mu_1^2)}\right)r_1r_2\right.\\
&\qquad\left.+\left(\frac{-11}{64}\right)r_2^2+\left(\frac{3\alpha^2\mu_2^2-2\alpha^2\mu_1^2}{8(\mu_1^2-\mu_2^2)}\right)r_1^2\right],\\
R_{21} &= R_{22} = 0.
\end{aligned}$$

Nayfeh [27] obtains a similar second order expansion for a single van der Pol oscillator.

From (4.5) and (4.8), the initial conditions for the first order approximation are taken as

$$(5.14)\qquad \phi_{10}(\varepsilon)=\theta_0, \qquad r_{10}(\varepsilon)=x_0,$$

where θ_0, x_0 are the initial conditions for (5.4). For the second order approximation the initial conditions are given by

$$(5.15)\qquad \phi_{20}(\varepsilon)=\theta_0-\varepsilon u_1(\theta_0, x_0), \qquad r_{20}(\varepsilon)=x_0-\varepsilon w_1(\theta_0, x_0).$$

From (4.13), the first order approximation is given by

$$(5.16)\qquad \theta_1(t,\varepsilon)=\phi_1(t,\varepsilon), \qquad x_1(t,\varepsilon)=r_1(t,\varepsilon),$$

where

$$(5.17)\qquad \begin{aligned}
\dot{\phi}_1 &= d+\varepsilon\Phi_1(r_1), \qquad \phi_1(0,\varepsilon)=\phi_{10}(\varepsilon),\\
\dot{r}_1 &= \varepsilon R_1(r_1), \qquad r_1(0,\varepsilon)=r_{10}(\varepsilon).
\end{aligned}$$

The second order approximation is given by

$$(5.18)\qquad \begin{aligned}
\theta_2(t,\varepsilon) &= \phi_2(t,\varepsilon)+\varepsilon u_1(\phi_2(t,\varepsilon), r_2(t,\varepsilon)),\\
x_2(t,\varepsilon) &= r_2(t,\varepsilon)+\varepsilon w_1(\phi_2(t,\varepsilon), r_2(t,\varepsilon)),
\end{aligned}$$

where

$$(5.19)\qquad \begin{aligned}
\dot{\phi}_2 &= d+\varepsilon\Phi_1(r_2)+\varepsilon^2\Phi_2(r_2), \qquad \phi_2(0,\varepsilon)=\phi_{20}(\varepsilon),\\
\dot{r}_2 &= \varepsilon R_1(r_2)+\varepsilon^2 R_2(r_2), \qquad r_2(0,\varepsilon)=r_{20}(\varepsilon).
\end{aligned}$$

In order to test the extent of application of Theorem 4.1, a simulation was performed on a computer that carried approximately 8 digits in single precision. A code using an Adams–Moulton procedure was executed using the following selection of equation parameters:

$$(5.20)\qquad \begin{aligned}
a &= 0.1250, \qquad \mu_1 = 1.0000,\\
\alpha &= 0.1250, \qquad \mu_2 = 1.4142.
\end{aligned}$$

The initial conditions used were

$$(5.21)\qquad \theta_{10}=\theta_{20}=0.0, \qquad x_{10}=x_{20}=10.0.$$

In the two tables below the absolute errors are computed between the solution of (5.5) and the first order approximation (5.16) as well as between the solution of (5.5) and the second order approximation (5.18). The initial conditions for (5.18) are taken as (5.15), using (5.21).

Table 1 shows the maximum absolute error encountered with $\varepsilon = 0.001$, 0.01 and 0.1 for the first and second order approximations. Table 2 shows the maximum absolute error encountered with $\varepsilon = 0.001$ for 1000 and 10,000 time steps.

In Table 1 the time scale of 100 steps is $O(1/0.01)$, so that the errors in the first two columns should be consistent with the conclusion of Theorem 4.1. In the first

TABLE 1
Maximum absolute error for 100 *simulation time steps of* 1 *unit per step.*

	$\varepsilon = 0.001$	$\varepsilon = 0.01$	$\varepsilon = 0.1$
	First Order Approximation		
θ_1	0.00254	0.0198	0.1100
θ_2	0.00105	0.0102	0.0856
x_1	0.01254	0.1030	0.3830
x_2	0.00807	0.0570	0.3030
	Second Order Approximation		
θ_1	0.000290	0.000225	0.00897
θ_2	0.000292	0.000232	0.00953
x_1	0.000033	0.002215	0.05750
x_2	0.000033	0.001734	0.11600

order approximation the errors should be $O(\varepsilon)$, which in general they are. The errors for x_1 seem somewhat out of line at first but, since we do not have a means of adequately estimating the constant C_N in (4.14), a constant of order 10 for the x_1 term would explain the result. Even though $\varepsilon = 0.1$ is large for a time scale of 100 the absolute errors for $\varepsilon = 0.1$ are still consistent with the theorem. The second order approximations are also consistent, again noting that we do not have an adequate bound on C_N in (4.14). There is certainly an order of magnitude or greater improvement

TABLE 2
Maximum absolute error or $\varepsilon = 0.001$ *for two time step histories.*

	1000 steps	10,000 steps
	First Order Approximation	
θ_1	0.0322	2.234
θ_2	0.0317	2.234
x_1	0.00889	0.00325
x_2	0.00556	0.00238
	Second Order Approximation	
θ_1	0.0321	2.234
θ_2	0.0321	2.234
x_1	0.000213	0.00366
x_2	0.000156	0.00300

in the errors for $\varepsilon = 0.001$, 0.01. For $\varepsilon = 0.1$ there is improvement, if not an order of magnitude in all the variables. In Table 2 the errors for x_1, x_2 are consistent with $\varepsilon = 0.001$, but as would be expected the errors in θ_1, θ_2 grow as time progresses..

Neu [29] simulated the second order average of another coupled system. His results are also consistent with the conclusion that the approximations are good over a time interval of order $O(1/\varepsilon)$ but deteriorate afterwards. His particular asymptotic approximations are also computed in a similar manner to the general results obtained in this paper, although he was also concerned with eliminating secular terms since he did not assume a nonresonance condition as we did in (4.9). In general, though, the two methods are comparable.

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